Math 4200-001
Week 7-8 concepts and homework
2.4
Due Friday October 16 at 11:59 p.m.

2.4 2, 3, 5, 7, 8, 12, 16, 17, 18. Hint: In problems 2, 5, 18 identify the contour integrals as expressing a certain function or one of its derivatives, at a point inside γ, via the Cauchy integral formulas for analytic functions and their derivatives.

w7.1 Prove the special case of the Cauchy integral formula that we discuss on Wednesday, in Monday's notes:

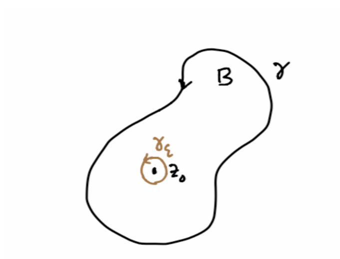
If γ is a counter-clockwise simple closed curve bounding a subdomain B in A, with z_0 inside γ, then the important special case of the Cauchy integral formula can be proven with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic:

$$\underbrace{I(\gamma; z_0)}_{=1}\, f(z_0) \overset{?}{=} \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\, dz. \overset{\text{replacement thm}}{=} \frac{1}{2\pi i}\oint_{|z-z_0|=\varepsilon} \frac{f(z)}{z-z_0}\, dz \quad *$$

$f(z) \to f(z_0)$ as $z \to z_0$

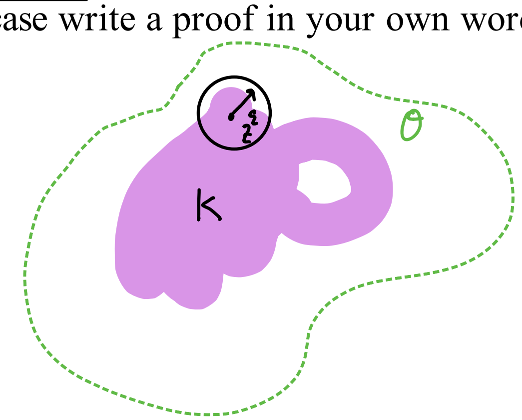
$$* \; \boxed{\approx} \; \underbrace{\frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} \frac{f(z_0)}{z-z_0}\, dz}_{\frac{f(z_0)}{2\pi i}\cdot 2\pi i \,=\, \boxed{f(z_0)}} + \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

to show this term $\to 0$ as $\varepsilon \to 0$



New today!!

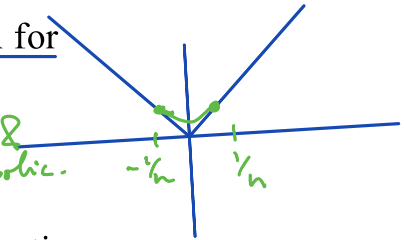
w7.2 Prove the positive distance lemma, which we make much use of in proving various theorems: If $K \subseteq \mathbb{C}$ is compact, and if $K \subseteq O$, where O is open, then there exists an $\varepsilon > 0$ such that for each $z \in K$, $D(z; \varepsilon) \subseteq O$. (This is equivalent to Distance Lemma 1.4.21 in the text. See if you can find a proof without looking there first, but in any case write a proof in your own words.)



two ways to approach
1) Use sequential compactness
proof by $\nRightarrow$
2) Pick a good cover, for which a finite subcover yields result

Corollary Let $\{f_n\}: A \to \mathbb{C}$ analytic. Suppose $\{f_n\} \to f$ uniformly on A. Then $f: A \to \mathbb{C}$ is also analytic. (Contrast this with the analogous false theorem for differentiable functions on subdomains of $\mathbb{R}$).

e.g. $f_n(x) \to |x|$ unif.
f_n C^1, p.w. linear & parabolic.



proof: Can you check these pieces, and combine them into a proof?

(i) f is continuous, because uniform limits of continuous functions are continuous. (3210-3220?)

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(ii) If $\{f_n\} \to f$ uniformly on A and if the rectangle lemma holds for each f_n (which it does, because each f_n is analytic), then the rectangle lemma holds for f.

(ii) f_n analytic $\Rightarrow$ Rect lemma holds ($\forall$ rect in $D(z_0;R) \subset A$)

$f_n \to f$ unif in A $\Rightarrow$ Rect lemma for f:

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$$\int_{\partial R} f(z)\,dz = \underbrace{\int_{\partial R} f_n(z)\,dz}_{\substack{f_n \text{ anal.} \\ 0}} + \int_{\partial R} f(z) - f_n(z)\,dz$$

$$| \quad\quad | \le \left| \int_{\partial R} f(z) - f_n(z)\,dz \right| \le \int_{\partial R} | \quad\quad | \, |dz| \quad *$$

$\forall\, \varepsilon > 0 \ \exists\, N$ s.t. $|f(z) - f_N(z)| < \varepsilon \quad \forall z \in A$

$$\le \int_{\partial R} \varepsilon\, |dz| = \varepsilon \cdot \text{perim of } R$$

ε arbitrary

$$\Rightarrow \left| \int_{\partial R} f(z)\,dz \right| = 0.$$

f sats rect lemma, Morera $\Rightarrow$ f analytic.

to be continued.

One of the most-studied analytic functions is the *Riemann -Zeta function*. It is customary to write the complex variable as s in this case, rather than z. And for $\mathrm{Re}(s) > 1$, the Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad , \quad \mathrm{Re}\, s > 1$$

where for $s = x + i\,y$, each term

$$n^{-s} = e^{-s\log(n)} = e^{-(x+iy)\ln(n)} = n^{-x} e^{-iy\ln(n)}$$

$1 \quad |\;| = 1$

is analytic in s. Note that for $x > 1$, the sum of moduli

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^x} < \infty$$

and for $x \ge 1 + \delta$ (with $\delta > 0$) the absolute convergence is uniform, so also the partial sums

$$\zeta_N(s) := \sum_{n=1}^{N} \frac{1}{n^s}$$

converge uniformly to $\zeta(s)$. Thus $\zeta(s)$ is analytic on the half plane $\mathrm{Re}(s) > 1$, by ~~Morera's Theorem~~. Your favorite divergent series

unif. conv thm

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$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \qquad \text{harmonic series}$$

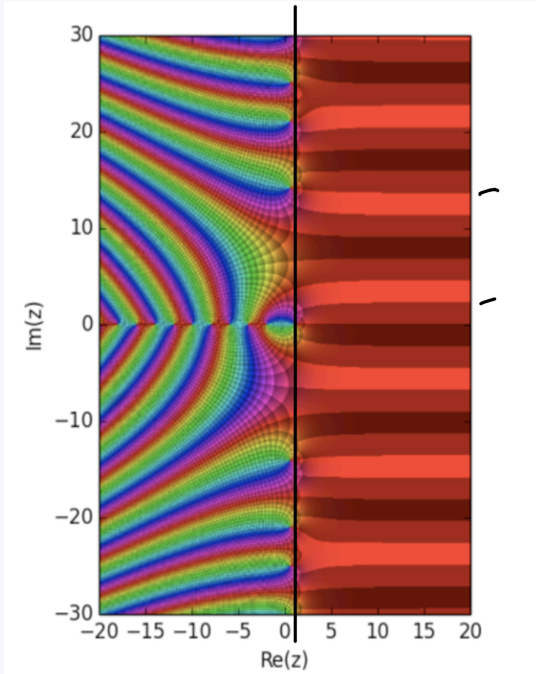
shows that $\zeta(s)$ is not analytic at $s = 1$. Somewhat surprisingly, $\zeta(s)$ can be extended to be analytic in all of $\mathbb{C} \setminus \{1\}$, however. (Such extensions must always be unique, it turns out.) The formulas for this extended function $\zeta(s)$ look different than the one that works on the half plane $\mathrm{Re}(s) > 1$.

The Riemann Zeta function has surprising connections to *number theory*, in particular to the *prime number theorem*, which is about how prime numbers are distributed in the natural numbers.
The *Riemann Hypothesis* is Riemann's conjecture from the 1800's, that all of the so-called non-trivial zeroes of the Riemann function lie on the line $\left\{ \mathrm{Re}(s) = \frac{1}{2} \right\}$. (The other zeroes of the zeta function occur at the negative even integers.) It's considered one of the greatest unproven conjectures in mathematics, see for example the *Millenium prizes*. Of the billions of zeroes of the Riemann function which have been found, they're all on that line! Many results in number theory would follow if the Riemann hypothesis is true, so people are in the habit of proving theorems, where one of the assumptions is that the Riemann Hypothesis is true.

This is a great topic area for a research report in our course, if your interests go in this direction.

The output of the zeta function, plotted as a "graph" above the complex domain, with contours for the modulus and so that the color represents the argument of $\zeta(z)$. From wikipedia:

Riemann zeta function



The Riemann zeta function $\zeta(z)$ plotted with domain coloring.[1]

Basic features

Domain	$\mathbb{C} \setminus \{1\}$
Codomain	$\mathbb{C}$

Specific values

At zero	$-\frac{1}{2}$
Limit to +∞	1
Value at 2	$\frac{\pi^2}{6}$
Value at -1	$-\frac{1}{12}$
Value at -2	0

$\mathrm{Re}\, s > 1$

$$\zeta(x+iy) = \sum_{n=1}^{\infty} \frac{1}{n^{x+iy}}$$

$\mathrm{Re}\, s \gg 1$

$$1 + \frac{1}{2^{x+iy}} + \cdots$$

$$\frac{1}{2^x} \quad \frac{1}{e^{iy \ln 2}}$$

period in y-dir $\frac{2\pi}{\ln 2} \approx 10$

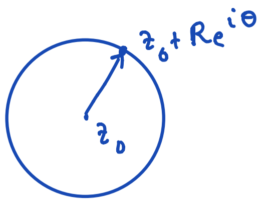
Math 4200
Friday October 16
2.4-2.5 mean value property for analytic and harmonic functions, and maximum modulus principles. But We'll begin by finishing Wednesday's notes with the introduction to the Riemann-Zeta function. The mean value property and maximum principle have many consequences, as we'll see on Monday.

Announcements: HW questions?
w7.1 overview. (we wrote notes but
w7.2. I forgot to record this part).

Mean value property Let $f: A \to \mathbb{C}$ analytic, $\bar{D}(z_0; R) \subseteq A$. Then the value of f at z_0 is the average of the values of f on the concentric circle of radius R about z_0:

$$f(z_0) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + R\, e^{i\theta}\right) d\theta}_{\text{average value of } f \text{ on circle of rad } R \text{ ctrd at } z_0.}$$

proof:



C.I.F. for γ, z_0

$$f(z_0) = \frac{1}{2\pi i}\oint_{|z-z_0|=R} \frac{f(z)}{z-z_0}\,dz$$

$\gamma(\theta) = z_0 + Re^{i\theta}$ $0 \le \theta \le 2\pi$, $\gamma'(\theta) = iRe^{i\theta}$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{\cancel{Re^{i\theta}}}\,\cancel{iRe^{i\theta}}\,d\theta$$

$$\underbrace{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta \quad !}$$

Remark harmonic functions also satsify a mean value property. How do you think you'd go about proving it?

Let u be harmonic domain containing $\overline{D(z_0;R)}$

use local conjugate v

s.t. $u + iv = f$ is analytic

$$u(x_0,y_0) + iv(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta) + iv(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta$$

isolate real part

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta.$$

A cool way to justify the harmonic conjugate construction in simply connected domains. ...and also to deduce that harmonic functions are infinitely differentiable.....we appealed back to multivariable calculus earlier in the course for the conjugate construction. (Harmonic conjugates show up in our proof of the mean value property for harmonic functions, ~~at the end of Wednesday's notes.)~~
in previous page.

Theorem Let $A \subseteq \mathbb{C}$ be open and simply connected, and let $u : A \to \mathbb{R}$ be C^2 and harmonic. Then there exists a harmonic conjugate $v : A \to \mathbb{R}$, i.e. so that $f = u + i\,v$ is analytic. Furthermore, both u, v are actually C^∞, i.e. all partial derivatives exist and are continuous.

proof. If f existed, then f would be infinitely complex differentiable, and so in particular f' would be analytic...

$$\begin{aligned} f' &= f_x = u_x + i\,v_x \\ &\overset{CR}{=} v_y - i\,u_y. \end{aligned}$$

In other words,

$$g(z) = u_x - i\,u_y$$

would be analytic. Actually, CR holds for $g(z)$ defined as above just because u is harmonic and C^2, and because g has continuous first partials, so g IS analytic: Check (This was previous HW):

CR1 $(u_x)_x = (-u_y)_y$ $u_{xx} = -u_{yy}$ true u harmonic

CR2 $(u_x)_y = -(-u_y)_x = u_{yx}$ true for C^2 fcns.

Since g is analytic on A and A is simply connected, g has an antiderivative $G = U + i\,V$. $G' = g$ so

$$G'(z) = U_x + i\,V_x \overset{CR}{=} V_y - i\,U_y = u_x - i\,u_y = g$$

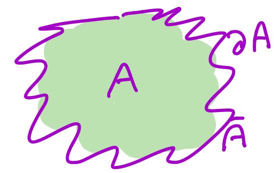
so $U_x = u_x$, $U_y = u_y$ so $U = u + C$ where C is a real constant because A is connected. Thus

$$f := G - C = u + i\,V \quad \text{is analytic \& } \infty\text{'ly diff'ble.}$$

is analytic, i.e. V is a harmonic conjugate to u. Since G is infinitely complex differentiable, u, V are infinitely real differentiable.

QED.

<u>Theorem</u> (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f: A \to \mathbb{C}$ be analytic, $f: \overline{A} \to \mathbb{C}$ continuous. Then



Part 1

$$\max_{z \in \overline{A}}\{|f(z)|\} = \max_{z \in \delta A}\{|f(z)|\} := M,$$

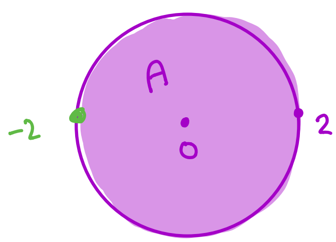
i.e. the maximum absolute value of $f(z)$ occurs on the boundary of A. (Recall that for an open set A, the boundary $\delta A = \overline{A} \setminus \mathring{A}$. For general sets the boundary is the collection of points which are in the closure of the set as well as in the closure of its complement.)

Part 2

Furthermore if $\exists\, z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function on A!

Example: What is the maximum absolute value of $f(z) = (z-2)^2$ on the closed disk $\overline{D}(0; 2)$ and where does it occur ?

where is $|z-2|^2$ largest on $\overline{D(0;2)}$?



$|z-2|$ is dist from z to 2.
maximized at $z=-2$.
max $|z-2|^2$ on closed disk is $4^2 = 16$.
occurs on bdry of disk

proof of maximum modulus principle: Let

$$B = \{z \in A \mid |f(z)| = M\}$$

to be continued!!

Our goal is to show that either:

(i) $B = \varnothing$, which implies that all points in $\overline{A}$ for which $|f(z)| = M$ are on the boundary of A, as the theorem claims. And in this case there is no $z_0 \in A$ with $|f(z_0)| = M$.

OR

(ii) $B = A$. In this case $|f(z)|$ is constant. Write $f = u + i\,v$ and so we have

$$u^2 + v^2 \equiv M^2$$

If $M = 0$ then $f = 0$ on A and we are done. Otherwise $M > 0$ and taking x and y partials we get the system for each $z \in A$:

$$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $M \neq 0$, $(u, v) \neq (0, 0)$ at any point. Thus the determinant of the matrix is identically zero. But the determinant of the matrix is

$$u_x v_y - u_y v_x = u_x^2 + u_y^2 = v_y^2 + v_x^2.$$

Thus the gradients of u, v are identically zero on the connected open set A, so u and v are each constants on A and f is as well. This must be the case that occurs if $\exists\ z_0 \in A$ with $|f(z_0)| = M$.

Following the outline on the previous page, we have

$$B = \{z \in A \mid |f(z)| = M\}.$$

Suppose we are not in case (i), i.e. $B \neq \varnothing$. We will show that B is open and closed in A which will imply that B must be all of A, since A is connected. Thus we are in case (ii).

Why is B closed in A?

To show B is open, let $z_0 \in B$, $D(z_0, \rho) \subseteq A$. We'll show $|f(z)| = M$ $\forall\, z \in D(z_0, \rho)$. Each such z in the disk is of the form $z = z_0 + r\,e^{i\theta}$ with $r < \rho$. But for $0 < r < \rho$ we have the mean value property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + r\,e^{i\theta}\right)\, d\theta.$$

Use this and $|f(z_0)| = M$ to show each $\left|f\left(z_0 + r\,e^{i\theta}\right)\right| = M$ as well.

This generalizes an exercise that is due today (2.4.4) where you assume a domain A was bounded by a simply connected, p.w. C^1 contour and probably use the CIF.

<u>Theorem</u>

Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f, g : A \to \mathbb{C}$ be analytic, $f, g : \overline{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \overline{A}} \{|f(z) - g(z)|\} = \max_{z \in \delta A} \{|f(z) - g(z)|\}.$$

In particular, if $f = g$ on δA, then $f = g$ on all of A.

proof:

Theorem (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u : A \to \mathbb{R}$ be harmonic and C^2, $u : \overline{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x,\,y)\,\in\,\overline{A}}\{u(x,y)\} = \max_{(x,\,y)\,\in\,\delta A}\{u(x,y)\} := M,$$
$$\min_{(x,\,y)\,\in\,\overline{A}}\{u(x,y)\} = \min_{(x,\,y)\,\in\,\delta A}\{u(x,y)\} := m,$$

Furthermore if $\exists\ (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then u is a constant function on A.

Example: $u(x, y) = x^2 - y^2$ is harmonic. Where are the maximum and minimum values of u attained, on $\overline{\mathrm{D}}(0; 2)$?

proof: The maximum principle implies the minimum principle, since the minimum principle for $u(x, y)$ is equivalent to the maximum principle for $v(x, y) = -u(x, y)$. In other words, minimum values for $u(x, y)$ correspond to maximum values for $-u(x, y)$, and u is harmonic if and only if $-u$ is. So we'll focus on the maximum principle. The key tool is the mean value principle for harmonic functions: For every closed disk in A, the average value of u on the bounding circle equals the value at the center. Can you see how the proof goes, if we follow the outline of the maximum modulus principle proof?

Use: if $\mathrm{D}(z_0; \rho) \subseteq A$ then for each $0 < r < \rho$,

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u\big(x_0 + r\cos(\theta),\, y_0 + r\sin(\theta)\big)\, d\theta$$

Math 4200-001

Week 8 concepts and homework

2.4-2.5

Due Friday October 23 at 11:59 p.m.

2.5 2, 5, 7, 8, 10, 15, 18.

3.1 4, 6, 7, 12